Towards a variational principle for motivated vehicle motion

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We deal with the problem of deriving the microscopic equations governing individual car motion based on assumptions about the strategy of driver behavior. We presume the driver behavior to be a result of a certain compromise between the will to move at a speed that is comfortable for him under the surrounding external conditions, comprising the physical state of the road, the weather conditions, etc., and the necessity to keep a safe headway distance between the cars in front of him. Such a strategy implies that a driver can compare the possible ways of further motion and so choose the best one. To describe the driver preferences, we introduce the priority functional whose extremals specify the driver choice. For simplicity we consider a single-lane road. In this case solving the corresponding equations for the extremals we find the relationship between the current acceleration, velocity, and position of the car. As a special case we get a certain generalization of the optimal velocity model similar to the ''intelligent driver model'' proposed by Treiber and Helbing.

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I. INTRODUCTION

The fundamentals of traffic flow dynamics are far from being well established because up to now, what the specific form of the microscopic equations governing the individual car motion should be has not been clear. The problem is that the car motion is controlled by the motivated driver behavior rather than obeying Newton's laws. In fact, the behavior of a driver is due to a certain compromise between the will to move at a speed that is comfortable for him and attained on an empty road on one hand, and the need to avoid possible traffic accidents on the other. So, comparing car ensembles with physical systems, it is not obvious beforehand that there is a direct relationship between the acceleration of a given car and the positions and the velocities of other cars as is the case for physical particles. In addition, the linear superposition typical in the interaction of physical particles is not self-evident in the case of vehicle interaction.

By contrast, on macroscopic scales the car ensembles exhibit a wide class of critical and self-organization phenomena widely met in physical systems (for a review see Refs. [1–4]). It should be pointed out that fish and bird swarms, colonies of bacteria, pedestrians, etc. also demonstrate similar cooperative motion (for a review see Ref. [4] and also Refs. [5–7]). So the cooperative behavior of many-particle systems, including social and biological ones, seems to be of a more general nature than the mechanical laws, and determining what microscopic regularities are responsible for the cooperative phenomena in the general case is a challenging problem. Traffic dynamics is widely studied in this context also due to the great potential for industrial applications.

The currently adopted approach to specifying the microscopic governing equations of the individual car motion is the so-called social force model, or generalized force model. Its detailed motivation and description can be found in Refs. [8–10]; here we only touch on the basic ideas. At each moment t of time, a given driver α increases or decreases the speed v_α of his car or keeps its value unchanged depending on the road conditions and the arrangement of the neighboring cars:

$$\frac{dv_\alpha}{dt}=f_\alpha(v_\alpha)+\sum_{\alpha'\neq\alpha} f_{\alpha\alpha'}(x_\alpha,v_\alpha|x_{\alpha'},v_{\alpha'}). \tag{1.1}$$

Here the term $f_\alpha(v_\alpha)$, typically of the form

$$f_\alpha(v_\alpha)=\frac{v_\alpha^0-v_\alpha}{\tau_\alpha},$$

describes the driver tendency to move on the empty road at a certain fixed speed v_α^0 depending on the physical state of the road, weather conditions, legal traffic regulations, etc. The relaxation time τ_α characterizes the acceleration capability of the given car as well as the delay in the driver control over the headway. The term $f_{\alpha\alpha'}(x_\alpha,v_\alpha|x_{\alpha'},v_{\alpha'})$ describes the interaction of car α with car α' ($\alpha'\neq\alpha$), which is due to the necessity for driver α to keep a certain safe headway distance between the cars. The force $f_{\alpha\alpha'}(x_\alpha,v_\alpha|x_{\alpha'},v_{\alpha'})$ is assumed to depend directly only on the velocities v_α, $v_{\alpha'}$ and the positions x_α, $x_{\alpha'}$ of this car pair, and being of a nonphysical nature does not meet Newton's third law, i.e., in the general case $f_{\alpha\alpha'}\neq -f_{\alpha'\alpha}$. Reference [11] presents and discusses possible generalized *Ansätze* for the dependence $f_{\alpha\alpha'}(x_\alpha,v_\alpha|x_{\alpha'},v_{\alpha'})$.

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The special cases of this model have their own names. In particular, for a single-lane road, when all the cars can be ordered according to their position on the road in the car motion direction,

$$\cdots < x_{\alpha-2} < x_{\alpha-1} < x_\alpha < x_{\alpha+1} < x_{\alpha+2} < \cdots,$$

the interaction solely of the nearest neighboring cars α and $\alpha+1$ is taken into account, i.e., is $f_{\alpha\alpha'} \neq 0$ for $\alpha' = \alpha+1$ and, may be, $\alpha' = \alpha-1$ only. For this case Bando *et al.* [12,13] proposed the optimal velocity model that describes the individual car motion as

$$\frac{dv_\alpha}{dt} = \frac{1}{\tau}[\vartheta_{\text{opt}}(x_{\alpha+1} - x_\alpha) - v_\alpha], \tag{1.2}$$

where $\vartheta_{\text{opt}}(\Delta)$ is the steady-state velocity (the optimal velocity) chosen by drivers for the given headway distance $\Delta = x_{\alpha+1} - x_\alpha$ between the cars. This model and its modification were successfully used to explain the properties of the "stop-and-go" waves that develop in dense traffic on single-lane roads (see, e.g., Refs. [14–30]).

However, on multilane highways the behavior of traffic flow becomes sufficiently complex because of the strong correlations in the car motion in different lanes (for a review see Refs. [1–4]). In this case it is not sufficient to confine the consideration only to the interaction between the nearest neighboring cars and one has to specify several independent components of the social forces $\{f_{\alpha\alpha'}(x_\alpha, v_\alpha | x_{\alpha'}, v_{\alpha'})\}$. As a result, the number of essential fitting parameters entering the social forces increases substantially. A possible way to overcome this problem is to formulate a mathematical principle characterizing the strategy of driver behavior in terms of a certain functional quantifying the objectives pursued by drivers. Constructing this functional may be, at the phenomenological level, made easier by the clear physical meaning of the driver's objectives. Then, using standard techniques one will derive the required governing equations. This work presents our first steps towards such an approach. It should be noted that the derivation of microscopic governing equations for systems with motivated behavior based on a certain "optimal self-organization" principle was discussed in Refs. [4,7,31–33]. The main idea of this approach is the assumption that individuals try to minimize the interaction strength or, in other words, to optimize their own success and to minimize the efforts required for this.

Our approach is related to the concepts of mathematical economics, namely, to the concepts of preferences and utility (see, e.g., Ref. [34]). We suppose that at each moment t_0 of time, a driver plans his motion in a certain way in order, first, to move as fast as possible and, second, to prevent traffic accidents. In particular, in the previous paper [35], using this idea we developed a model explaining in a simple way the experimentally observed sequence of the first-order phase transitions from the "free flow" to the jam phase through the "synchronized mode" [36–38]. Such a strategy actually implies that a driver evaluates any possible path of his further motion, $\{\chi(t),\ t > t_0\}$, with respect to its preferability. In other words, a driver can compare any two paths $\{\chi_1(t),\ t > t_0\}$ and $\{\chi_2(t),\ t > t_0\}$ and decide which of them is more preferable, for example, $\chi_1(t)$. The latter relation will be designated as $\chi_1(t) \succ \chi_2(t)$. Obviously, the given relation exhibits transitivity, i.e., if $\chi_1(t) \succ \chi_2(t)$ and $\chi_2(t) \succ \chi_3(t)$ then $\chi_1(t) \succ \chi_3(t)$. In this case we may seek a priority functional $\mathcal{L}\{\chi\}$ meeting the condition $\mathcal{L}\{\chi_1\} > \mathcal{L}\{\chi_2\}$ when and only when $\chi_1 \succ \chi_2$. Finally the driver chooses the best path $\chi_{\text{opt}}(t)$ of his further motion maximizing the priority functional $\mathcal{L}\{\chi\}$. So its extremals have to satisfy the desired microscopic governing equation of individual car motion. It should be noted that the chosen path $\chi_{\text{opt}}(t)$ of the planned motion specifies the acceleration at the current time moment t_0 rather than the real trajectory $x(t)$ of the car motion because at the next time $t > t_0$ the driver again plans his motion in the same way, introducing the corrections caused by changes in the surroundings. The same concerns the car velocity and its acceleration. Therefore to avoid possible misunderstanding we will designate the real velocity and acceleration of the car as $v(t)$ and $a(t)$, whereas the values corresponding to the optimal $\chi_{\text{opt}}(t)$ will be labeled by $\{\nu(t),\ t > t_0\}$ and $\{\varpi(t),\ t > t_0\}$, respectively.

In this way the problem of specifying many independent components of the social forces is solved by constructing the priority functional describing the driver compromise between the will to move as fast as the physical state of the road allows and the necessity to avoid possible traffic accidents. So to obtain the priority functional we may apply general assumptions about driver behavior.

II. VARIATIONAL PRINCIPLE FOR THE INDIVIDUAL CAR MOTION

First we should determine the collection of phase variables characterizing the quality of a given car motion. We note that for the driver under consideration the neighboring car arrangement and its evolution should be regarded as given beforehand. Indeed, it cannot be directly controlled by him and so has to be treated as an external condition. Formulating this problem we actually assume the existence of a certain collection of variables *taken at the current time moment t* that completely quantify the priority measure of the car motion at the same time. Adopting the latter assumption, we may construct the priority functional $\mathcal{L}\{\chi\}$ in terms of a certain integral of a function $\mathcal{F}$ of the phase variables with respect to time.

Using conventional driver experience, we will characterize the individual car motion at each time moment t by its position on the road $x(t)$, the velocity $v(t)$, and the acceleration $a(t)$. For a multilane highway, for example, the position of a car should also bear information about the lane occupied by the car, but this problem will be considered elsewhere and in the present paper we confine our consideration to a single-lane road only. Due to this property, a car is distinct from a physical particle because the motion of the latter is completely determined by its current position and velocity. The variables $x(t)$, $v(t)$, and $a(t)$, however, exhibit different behavior. The coordinate $x(t)$ and the velocity $v(t)$ of the car vary continuously, i.e., the driver cannot change them immediately. In contrast, the acceleration $a(t)$ may exhibit sharp jumps because it is the acceleration that is controlled directly by the driver without significant delay. In such an analysis it is quite reasonable to ignore the short physiological delay in the driver's behavior to changes in the surroundings, allowing sharp jumps in the dependence $a(t)$. Therefore, planning his further path of motion the driver regards the position $x_0 := x(t_0)$ and the velocity $v_0 := v(t_0)$ of the car at the current moment of time t_0 as the initial data.

Now let us write the general form of the priority functional $\mathcal{L}\{\chi\}$ for a trial path $\{\chi(t),\ t > t_0\}$ of the further motion,

$$\mathcal{L}\{\chi\} = -\int_{t_0}^{\infty} dt \exp\left(-\frac{\chi - x_0}{\ell}\right)\mathcal{F}(t,\chi,\nu,\varpi), \qquad (2.1)$$

where $\mathcal{F}(t,\chi,\nu,\varpi)$ is the density of the path priority measure, $\nu := d\chi/dt$ and $\varpi := d^2\chi/dt^2$, and the exponential cofactor reflects the fact that drivers can monitor the traffic flow state and so plan the motion only inside a certain region of length ℓ in front of them. Under normal conditions this region should enable a driver to govern his motion effectively, for example, to decelerate in advance, avoiding a possible accident. Therefore, its size ℓ has to meet the inequality $\ell \gtrsim \bar{v}\tau$, where $\bar{v}$ is the characteristic vehicle velocity in the current traffic flow and the time τ specifies the acceleration-deceleration capability of the given car. In what follows we will assume this inequality to hold. Besides, in expression (2.1) the leading minus has been chosen so as to reduce the problem of finding the maximum of the functional $\mathcal{L}\{\chi\}$ to that of determining the minimum of integral (2.1), as is the typical case in physical theories. The direct dependence of the function $\mathcal{F}(t,\chi,\nu,\varpi)$ on time t reflects the effect of the surroundings, i.e., the physical road state and the neighboring car arrangement, on the driver planning.

According to the adopted assumption, the driver chooses the path $\chi_{\rm opt}(t)$ of further motion that maximizes the functional $\mathcal{L}(t,\chi,\nu,\varpi)$, and at the current time t_0 together with other all trial paths $\{\chi(t)\}$ meets the conditions

$$\chi(t_0) = x_0, \qquad \nu(t_0) = v_0. \qquad (2.2)$$

Besides, the present paper analyzes car motion in traffic flow, i.e., it does not consider any means by which a fixed car can leave the traffic flow, for example, to stop. So we assume that all the trial paths exhibit bounded variations, i.e., there are constants C_χ^l, C_χ^u, C_ν, and C_ϖ such that

$$C_\chi^l (t - t_0) < \chi(t) < C_\chi^u (t - t_0),$$

$$\nu(t) < C_\nu, \qquad |\varpi(t)| < C_\varpi. \qquad (2.3)$$

Then, using the standard technique, we get the governing equation for the extremals of the priority functional $\mathcal{L}\{\chi\}$, following from the condition $\delta\mathcal{L}\{\chi\} = 0$ at $\chi(t) = \chi_{\rm opt}(t)$,

$$\frac{d^2}{dt^2}\left\{\exp\left[-\frac{\chi}{\ell}\right]\frac{\mathcal{F}(t,\chi,\nu,\varpi)}{\partial\varpi}\right\} - \frac{d}{dt}\left\{\exp\left[-\frac{\chi}{\ell}\right]\frac{\partial\mathcal{F}(t,\chi,\nu,\varpi)}{\partial\nu}\right\} + \frac{\partial}{\partial\chi}\left\{\exp\left[-\frac{\chi}{\ell}\right]\mathcal{F}(t,\chi,\nu,\varpi)\right\} = 0. \qquad (2.4)$$

By virtue of Eq. (2.3) the function $\mathcal{F}(t,\chi,\nu,\varpi)$ exhibits bounded variations, which enables us to integrate Eq. (2.4) twice with respect to time t, reducing it to the following:

$$\begin{aligned}\frac{\partial\mathcal{F}(t,\chi(t),\nu(t),\varpi(t))}{\partial\varpi(t)} &= -\int_t^{\infty} dt' \exp\left[-\frac{\chi(t') - \chi(t)}{\ell}\right] \\ &\quad\times \frac{\partial\mathcal{F}(t',\chi(t'),\nu(t'),\varpi(t'))}{\partial\nu(t')} - \int_t^{\infty} dt' \int_{t'}^{\infty} dt'' \frac{\partial}{\partial\chi(t'')} \\ &\quad\times\left\{\exp\left[-\frac{\chi(t'') - \chi(t)}{\ell}\right]\mathcal{F}(t'',\chi(t''),\nu(t''),\varpi(t''))\right\}.\end{aligned} \qquad (2.5)$$

Equation (2.5) relates the planned acceleration ϖ to the car position χ and velocity ν. Subjecting this equation to the initial conditions (2.2), we can find the optimal path $\chi_{\rm opt}(t|x_0,v_0)$ depending on the initial car position x_0 and velocity v_0. Then differentiating $\chi_{\rm opt}(t|x_0,v_0)$ twice with respect to t and setting $t = t_0$, we will get the desired relationship between the real current position $x_0 = x(t_0)$ and the velocity $v_0 = v(t_0)$ of the car with the acceleration $a(t_0)$ that the driver elects under these conditions. In other words, the expression obtained in such a way, i.e.,

$$a(t_0) = \lim_{t \to t_0 + 0} \frac{\partial^2 \chi_{\rm opt}(t|x_0,v_0)}{\partial t^2}, \qquad (2.6)$$

gives us the microscopic governing equation for the individual car motion. In addition, it should be noted that Eq. (2.4) is of fourth order, as it must be, because a trial path is fixed in part by the position and the velocity at the initial and terminal points. However, in the case under consideration the characteristics of the terminal point are replaced by conditions (2.3), allowing us to reduce the order.

Now let us demonstrate the proposed approach analyzing a simple example.

III. THE GENERALIZED OPTIMAL VELOCITY MODEL

In constructing the priority functional we assumed that in the steady-state traffic flow a driver prefers to move at a certain speed $\vartheta_{\rm opt}(t,x)$ depending on the surroundings and conditions given beforehand. Moreoever, we consider car motion without acceleration to be the best way of driving. Therefore, we adopt the following ansatz:

$$\mathcal{F}(t,x,v,a) = \frac{1}{2}[v - \vartheta_{\rm opt}(t,x)]^2 + \frac{1}{2}\tau^2 a^2, \qquad (3.1)$$

where the time scale τ characterizes the acceleration capability of the given car. The driver monitoring the car arrangement in front of him can predict the situation development, which is described in terms of the linear dependence of the optimal velocity $\vartheta_{\rm opt}(t,x)$ on time t and distance x,

$$\vartheta_{\rm opt}(t,x)=\vartheta^0_{\rm opt}\left[1+\varepsilon_t\frac{t-t_0}{\tau}+\varepsilon_x\frac{x-x_0}{\ell}\right], \tag{3.2}$$

where $\vartheta^0_{\rm opt}=\vartheta_{\rm opt}(t_0,x_0)$ and ε_t, ε_x are constants regarded here as small parameters of the same order. In addition, the difference $v(t)/\vartheta^0_{\rm opt}-1$ is also assumed to be of the order of $\varepsilon_t\sim\varepsilon_x$. We have adopted the linear dependence of $\vartheta_{\rm opt}(t,x)$ on t and x because it seems quite reasonable that a driver uses the linear approximation in estimating the position of the cars in front of him.

Substituting expressions (3.1) and (3.2) into formula (2.5) and truncating all the terms whose orders exceed $\varepsilon_t\sim\varepsilon_x\sim(v(t)/\vartheta^0_{\rm opt}-1)$ we get

$$\tau^2\varpi(t)+\int_t^\infty dt'\exp\left[-\frac{\vartheta^0_{\rm opt}(t'-t)}{\ell}\right][\nu(t')-\vartheta^0_{\rm opt}]$$

$$=\frac{\ell}{\tau\vartheta^0_{\rm opt}}(\varepsilon_t\ell+\varepsilon_x\tau\vartheta^0_{\rm opt})\left[1+\frac{\vartheta^0_{\rm opt}(t-t_0)}{\ell}\right]. \tag{3.3}$$

Multiplying Eq. (3.3) by the factor $\exp(-\vartheta^0_{\rm opt}t/\ell)$ and differentiating the obtained result with respect to t, we reduce Eq. (3.3) to the following for $t>t_0$:

$$\tau^2\frac{d^2\nu}{dt^2}-\sigma\tau\frac{d\nu}{dt}-\nu=-\vartheta^0_{\rm opt}\left[1+(\varepsilon_t+\sigma\varepsilon_x)\frac{(t-t_0)}{\tau}\right], \tag{3.4}$$

where we have introduced the parameter

$$\sigma=\frac{\tau\vartheta^0_{\rm opt}}{\ell}\lesssim 1,$$

the estimate of which results from the assumption adopted about the value of ℓ. The desired solution of Eq. (3.4) should meet the initial conditions (2.2) and situational conditions (2.3), which, within ansatz (3.2), convert to the requirement that dependence $\nu(t)$ not exhibit exponential growth. In this way we get

$$\nu(t)=v_0\exp\left[-\kappa\frac{(t-t_0)}{\tau}\right]+\vartheta^0_{\rm opt}\left\{[1-\sigma(\varepsilon_t+\sigma\varepsilon_x)]\right.$$

$$\left.\times\left(1-\exp\left[-\kappa\frac{(t-t_0)}{\tau}\right]\right)+(\varepsilon_t+\sigma\varepsilon_x)\frac{(t-t_0)}{\tau}\right\}, \tag{3.5}$$

where we have introduced the constant

$$\kappa=\left(\frac{1}{2}\sigma+\sqrt{\frac{1}{4}\sigma^2+1}\right)^{-1}\lesssim 1.$$

Substituting expression (3.5) into formula (2.6), we obtain the desired microscopic governing equation for the individual car motion:

$$\frac{dv}{dt}=-\frac{1}{\tau}\kappa\{v-\vartheta^0_{\rm opt}[1+\kappa(\varepsilon_t+\sigma\varepsilon_x)]\}, \tag{3.6}$$

where we have omitted the subscript 0 in the acceleration and velocity terms, implying that these values correspond to the current time moment. In particular, let the optimal velocity $\vartheta_{\rm opt}(t,x)=\vartheta_{\rm opt}(\Delta)$ be specified entirely by the headway distance $\Delta=x_{\alpha+1}-x_\alpha$ between the given car α and the nearest one $\alpha+1$ in front of it. Then within the linear approximation of situation development, the driver of car α can anticipate that the headway distance will change in time as

$$\Delta(t)=\Delta(t_0)+[v_{\alpha+1}(t_0)-v_\alpha(t_0)](t-t_0).$$

This expression together with the dependence $\vartheta_{\rm opt}(\Delta)$ enables us to calculate the specific value of the constant ε_t and then to rewrite formula (3.6) as

$$\frac{dv}{dt}=-\frac{1}{\tau}\kappa\left[v-\vartheta_{\rm opt}(\Delta)-\kappa\tau\delta v\frac{d\vartheta_{\rm opt}(\Delta)}{d\Delta}\right], \tag{3.7}$$

where we have introduced the relative velocity $\delta v=v_{\alpha+1}-v_\alpha$ of the car $\alpha+1$ with respect to the given car α and omitted the argument t_0, assuming all the values to be taken at the current moment of time.

It should be noted that the expression obtained (3.7) is similar to the phenomenological dependence $\vartheta_{\rm opt}(\Delta,\delta v)$ generalizing the standard optimal velocity model (1.2), the "intelligent driver model" proposed by Treiber and Helbing [39] (see also Ref. [40]).

IV. CONCLUSION

To conclude the present paper, we review its key points.

We deal with the problem of deriving microscopic equations governing the motion of individual vehicles. The currently adopted approaches similar to the social force model relate, in the spirit of Newton's laws, the acceleration of a given car to the position and velocities of the neighboring cars. In order to apply such models to the analysis of traffic dynamics, one has to specify all the essential components of the corresponding effective forces acting between the cars. However, when the vehicle interaction becomes sufficiently complex, as is the case, for example, for dense traffic on multilane highways, such an approach meets the problem of large numbers of fitting parameters.

The present paper proposes a possible way to avoid the aforementioned difficulty. The main idea is to describe at the first step the strategy of the driver behavior determined by the compromise between the driver's will to move as fast as possible on the given road, on one hand, and the necessity to keep a safe headway distance and not to interfere with cars moving in neighboring lanes, on the other hand. This assumption actually implies that a driver can compare various ways to proceed with respect to their relative advantages and choose the best (optimal) one at each time moment. This choice gives the relationship between the acceleration of the car under consideration and the arrangement of neighboring cars.

Following the concepts of mathematical economics, we have introduced a priority functional in order to quantify the driver choice. The extremals of this priority functional describe the optimal strategy by which the driver may proceed. In the present paper we have considered traffic flow on a single-lane road, constructed in this case the general form of the priority functional, and derived the equations for its extremals corresponding to car motion with traffic flow. The latter means that here we do not analyze the paths by which a car enters or leaves traffic flow on the given road because this question deserves individual investigation.

By way of example, we have considered a special case leading to an expression relating the current acceleration of a fixed car to the headway distance between this car and the one in front of it as well as their relative velocities. The equation obtained turns out to be similar to the ''intelligent driver model'' by Treiber and Helbing.

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